

BRS and Anti-BRS Symmetry in Topological Yang–Mills Theory

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Abstract

We incorporate both BRS symmetry and anti-BRS symmetry into the quantisation of topological Yang–Mills theory. This refines previous treatments which consider only the BRS symmetry. Our formalism brings out very clearly the geometrical meaning of topological Yang–Mills theory in terms of connections and curvatures in an enlarged superspace; and its simple relationship to the geometry of ordinary Yang–Mills theory. We also discover a certain $SU(3)$ triality between physical spacetime, and the two ghost directions of superspace. Finally, we demonstrate how to recover the usual gauge-fixed topological Yang–Mills action from our formalism.

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1. Motivation

Gauge theories apparently form the basis of fundamental physics. Electroweak theory and QCD are examples of Yang–Mills gauge theories associated with non-Abelian Lie groups. Even general relativity may be regarded, in a certain sense, as a gauge theory of the Lorentz group.

The key property of such a gauge theory is that its so-called gauge fields transform covariantly under transformations generated by a certain group. When one quantises the classical gauge theory using the Feynman path integral formalism, one has to integrate over all gauge fields. However because of this gauge-invariance, one is summing over redundant degrees of freedom, thus leading to infinite results. As it turns out, following the work of Feynman [1], DeWitt [2], Faddeev and Popov [3], and others, a way to evaluate properly the path integral is to first fix the gauge of the action, and then compensate for this breaking of gauge-invariance by introducing a Jacobian-like determinant to the measure of the path integral. This term, popularly known as the Faddeev–Popov determinant, can be written as a path integral over new anti-commuting fields called ghosts. It is these unphysical fields which ensure that the resulting quantum theory is gauge-invariant and unitary.

It was quite by accident when Becchi, Rouet and Stora [4] discovered a new set of transformations which leaves the full Faddeev–Popov Lagrangian invariant. This so-called BRS symmetry contains the original gauge symmetry, and it has since then been realised that it plays a fundamental rôle in quantum gauge theories, analogous to the rôle of gauge symmetry in the classical theory. This has led to our better understanding of the quantisation of gauge theories. Not only does “BRS-quantisation” simplify the heuristic process of Faddeev–Popov gauge-fixing, it also generalises to situations where the latter scheme breaks down. Furthermore, it provides a geometrical picture of the quantum gauge theory, not unlike the fibre bundle interpretation of classical gauge theories.

A good review of the modern aspects of the BRS symmetry in quantum gauge theories may be found in ref. [5]. In fact, our present day understanding of this symmetry also includes the so-called anti-BRS symmetry, discovered soon after the BRS symmetry as an additional symmetry of the Faddeev–Popov Lagrangian. An important result proved in ref. [5] is that, given any gauge theory whose infinitesimal transformations build up a closed algebra with a Jacobi identity, one can always construct the corresponding BRS

and anti-BRS symmetries. These are generated respectively by the BRS and anti-BRS operators s and \bar{s} , which satisfy the fundamental nilpotency condition

$$s^2 = \bar{s}^2 = 0 ; \tag{1a}$$

and anti-commute with each other:

$$s\bar{s} + \bar{s}s = 0 . \tag{1b}$$

Thus, it is important to realise that in order to completely characterise any quantum version of a gauge theory, both the BRS symmetry and the anti-BRS symmetry must be taken into account on an equal footing.

The particular type of gauge theory that we will be interested in, in this paper, is topological Yang–Mills theory, whose quantum theory was first modelled and shown by Witten [6] to generate the Donaldson invariants of smooth four-manifolds. The classical action of this theory is, for any compact gauge group G ,

$$\int_M \text{tr} [F \wedge F] , \tag{2}$$

where $F = dA + A \wedge A$ is the usual Yang–Mills field strength of the gauge potential one-form A . The trace is over the gauge group indices of F , and M is a compact four-manifold. The action is invariant under arbitrary variations of the gauge field A , and hence describes a topological field theory [7].

Baulieu and Singer [8], amongst others [9,10], have demonstrated how to BRS-quantise this classical action, resulting in Witten’s quantum gauge-fixed action. In fact, the BRS-quantisation scheme is presently the only known way to construct the quantum theory of (2), because of its peculiarly large gauge symmetry. It turns out that three ghost fields (together with their three associated anti-ghost fields and three Lagrange multiplier fields) are needed to completely break the symmetry. All the fields occurring in topological quantum Yang–Mills theory, and their properties, are listed in Table 1.

Full details of the construction of this BRS symmetry may be found in ref. [8]. Throughout most of this paper, we will adhere to the use of differential forms to describe the fields. Thus, A is an anti-commuting field in the sense that it is a one-form, while the two-form F is even. Also recall from ref. [8] that $\bar{\chi}$ and B are both self-dual two-forms, by choice of gauge-fixing.

Table 1. The fields of topological Yang–Mills theory

Field	Meaning	Form Degree	Ghost Number	Statistics
A	gauge potential	1	0	odd
F	field strength	2	0	even
c	usual Faddeev–Popov ghost	0	+1	odd
\bar{c}	anti-ghost of c	0	−1	odd
b	Lagrange multiplier	0	0	even
ψ	topological ghost	1	+1	even
$\bar{\chi}$	anti-ghost of ψ	2	−1	odd
B	Lagrange multiplier	2	0	even
ϕ	ghost for ghost ψ	0	+2	even
$\bar{\phi}$	anti-ghost of ϕ	0	−2	even
$\bar{\eta}$	Lagrange multiplier	0	−1	odd

Looking back at Table 1 again, a few asymmetries should catch the reader’s eye. Firstly, notice that the topological ghost ψ is a one-form, while its anti-ghost $\bar{\chi}$ is a self-dual two-form. Naively, one would expect the anti-ghost to have the same form degree as its corresponding ghost field, just as in the c – \bar{c} and ϕ – $\bar{\phi}$ systems. The other eye-sore is that the Lagrange multiplier field $\bar{\eta}$ having ghost number −1, is quite without a counterpart with ghost number +1 and the same form degree. Why should this be the case?

We claim that these asymmetries appear because only the BRS symmetry, and not the anti-BRS symmetry, has been built into the topological Yang–Mills theory. This is perhaps not too surprising a reason, in view of our remarks earlier in the paper, that the anti-BRS symmetry necessarily coexists with the BRS symmetry. In this paper, we will introduce both the BRS and anti-BRS symmetry into topological Yang–Mills theory, and recover Witten’s action just as Baulieu and Singer did using only the BRS symmetry. This is not just an unnecessary and pedagogical exercise. Apart from resolving the asymmetries noted above, it would also bring into full glory, the geometrical meaning of topological Yang–Mills theory in terms of connections and curvatures in an enlarged superspace. Furthermore, in this formalism, the relationship between the topological and the ordinary Yang–Mills theories would become so simple, it would seem hard to believe that topological Yang–Mills

theory had not been discovered earlier within the context of BRS and anti-BRS symmetry of ordinary Yang–Mills theory.

2. Construction of anti-BRS symmetry

Recall that the BRS symmetry that Baulieu and Singer [8] constructed rests upon the two gauge fields A and F ; and the three ghost fields c , ψ and ϕ . The action of the BRS operator s on these five fundamental fields is given by

$$\begin{aligned} sA &= \psi - Dc , \\ sc + \frac{1}{2}[c, c] &= \phi , \\ s\psi + [c, \psi] &= -D\phi , \\ s\phi + [c, \phi] &= 0 , \\ sF + [c, F] &= -D\psi , \end{aligned} \tag{3}$$

where $[,]$ is understood to mean the graded bracket, and $D \equiv d + [A, \cdot]$ is the gauge covariant derivative. This set is further supplemented by the s -transformations on their associated anti-ghosts and Lagrange multiplier fields:

$$\begin{aligned} s\bar{c} &= b , & sb &= 0 , \\ s\bar{\chi} &= B , & sB &= 0 , \\ s\bar{\phi} &= \bar{\eta} , & s\bar{\eta} &= 0 . \end{aligned} \tag{4}$$

Note that s raises the ghost number of its operand by $+1$, and it anti-commutes with d . It is also easy to verify from these equations that $s^2 = 0$.

Let us now postulate the existence of an anti-BRS operator \bar{s} with ghost number -1 ; and in addition to the above five fundamental fields, three anti-ghost fields \bar{c} , $\bar{\psi}$ and $\bar{\phi}$, corresponding to the ghost fields c , ψ and ϕ respectively. These anti-ghosts have the same form degree as their corresponding ghosts, but have ghost numbers that are opposite in sign. Then by an obvious mirror symmetry to the s -transformations (3), we demand that the following \bar{s} -transformations hold:

$$\begin{aligned} \bar{s}A &= \bar{\psi} - D\bar{c} , \\ \bar{s}\bar{c} + \frac{1}{2}[\bar{c}, \bar{c}] &= \bar{\phi} , \\ \bar{s}\bar{\psi} + [\bar{c}, \bar{\psi}] &= -D\bar{\phi} , \end{aligned} \tag{5}$$

$$\begin{aligned}\bar{s}\bar{\phi} + [\bar{c}, \bar{\phi}] &= 0 , \\ \bar{s}F + [\bar{c}, F] &= -D\bar{\psi} .\end{aligned}$$

Similar to the s case, we have that $\bar{s}^2 = 0$. Furthermore, we want s and \bar{s} to anti-commute, that is $s\bar{s} + \bar{s}s = 0$. In fact, one can easily verify that

$$\begin{aligned}(s\bar{s} + \bar{s}s)A &= s\bar{\psi} + \bar{s}\psi + [c, \bar{\psi}] + [\bar{c}, \psi] + D(s\bar{c} + \bar{s}c + [c, \bar{c}]) , \\ (s\bar{s} + \bar{s}s)F &= D(s\bar{\psi} + \bar{s}\psi + [c, \bar{\psi}] + [\bar{c}, \psi]) + [F, s\bar{c} + \bar{s}c + [c, \bar{c}]] .\end{aligned}\tag{6}$$

In order to make both of these expressions vanish, we must have as the most general possibility, that

$$\begin{aligned}s\bar{c} + \bar{s}c + [c, \bar{c}] &= \lambda , \\ s\bar{\psi} + \bar{s}\psi + [c, \bar{\psi}] + [\bar{c}, \psi] &= -D\lambda ,\end{aligned}\tag{7}$$

where λ is an even scalar field which has vanishing ghost number. The action of s and \bar{s} on λ may be derived by imposing the condition that $s\bar{s} + \bar{s}s$ acting on c and \bar{c} vanishes:

$$\begin{aligned}(s\bar{s} + \bar{s}s)c &= s\lambda + [c, \lambda] + \bar{s}\phi + [\bar{c}, \phi] = 0 , \\ (s\bar{s} + \bar{s}s)\bar{c} &= \bar{s}\lambda + [\bar{c}, \lambda] + s\bar{\phi} + [c, \bar{\phi}] = 0 .\end{aligned}\tag{8}$$

Having made these observations, it is merely routine to check that $s\bar{s} + \bar{s}s$ annihilates all the other fields, so that (1) is valid.

To summarise, we have so far identified nine fields, associated with a closed BRS and anti-BRS symmetry, and whose generators s and \bar{s} are nilpotent. It is interesting to plot the form degree of these nine fields against their ghost numbers, in which results in a suggestive pattern, as in Fig. 1. For now we take \mathcal{D} to generically denote an operator which increases by +1 the form degree, and thus acts in the upward direction. Analogously, \mathcal{S} and $\bar{\mathcal{S}}$ are operators which act toward the right and left respectively, raising and lowering the ghost number by one unit. We will say a few words about the significance of this pattern later on.

3. Geometrical interpretation

Let us now introduce an even-odd grading of the fields, according to whether the form degree plus ghost number of the field is even or odd. So our odd fields are A , c and \bar{c} ; while the remaining six fields F , ψ , $\bar{\psi}$, ϕ , $\bar{\phi}$ and λ are even. Note that the operators d , s and \bar{s} are all odd. This grading generalises that of ref. [8], which is simply form degree plus ghost number; sufficient to classify the fields A , c , F , ψ and ϕ only.

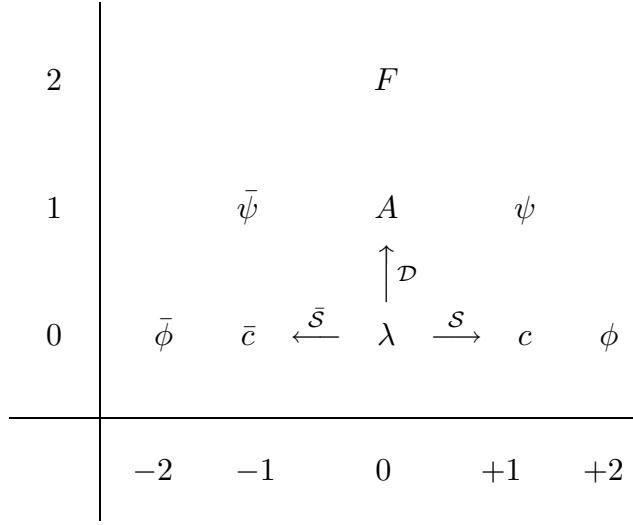


Fig. 1 Plot of form degree vs. ghost number

Since fields having the same grading are considered to be essentially of the same nature, we can add them together. Consider the two expressions

$$\begin{aligned}
 (d + s + \bar{s})(A + c + \bar{c}) + \frac{1}{2}[A + c + \bar{c}, A + c + \bar{c}] \\
 = F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda, \quad (9a)
 \end{aligned}$$

$$\begin{aligned}
 (d + s + \bar{s})(F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda) \\
 + [A + c + \bar{c}, F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda] = 0. \quad (9b)
 \end{aligned}$$

Upon expanding these equations out and collecting terms in form degree and ghost number, we recover all of the equations (3), (5), (7) and (8). That all these equations may be expressed so compactly in the two equations of (9) is not just a lucky coincidence. But how can we appreciate the significance of this?

The key [5] is to enlarge spacetime $\{x^\mu\}$ into a superspace \mathcal{M} , with two additional unphysical, anti-commuting coordinates θ and $\bar{\theta}$ at each point x^μ . Thus, \mathcal{M} has local coordinates $\{x^\mu, \theta, \bar{\theta}\}$, and we can proceed to define differential forms over this space. The generalised Yang–Mills gauge potential may be written as the one-form

$$\tilde{\mathcal{A}}(x, \theta, \bar{\theta}) = \tilde{A}_\mu(x, \theta, \bar{\theta})dx^\mu + \tilde{A}_\theta(x, \theta, \bar{\theta})d\theta + \tilde{A}_{\bar{\theta}}(x, \theta, \bar{\theta})d\bar{\theta}. \quad (10)$$

We will be only interested in fields restricted to the physical plane, whereby $\theta = \bar{\theta} = 0$. Such a field will be written without the tilde on the top. Observe that we can make the

identifications

$$A = A_\mu dx^\mu , \quad c = A_\theta d\theta , \quad \bar{c} = A_{\bar{\theta}} d\bar{\theta} , \quad (11)$$

so that the generalised Yang–Mills gauge potential over the physical spacetime is

$$\mathcal{A} = A + c + \bar{c} , \quad (12)$$

precisely the combination of fields occurring in (9). This means that the fields c and \bar{c} can be interpreted as components of the gauge potential \mathcal{A} in the unphysical directions θ and $\bar{\theta}$ respectively.

We can also define the analogue of the usual spacetime exterior derivative by

$$\tilde{d} = d + s + \bar{s} , \quad (13)$$

where

$$d \equiv dx^\mu \frac{\partial}{\partial x^\mu} , \quad s \equiv d\theta \frac{\partial}{\partial \theta} , \quad \bar{s} \equiv d\bar{\theta} \frac{\partial}{\partial \bar{\theta}} . \quad (14)$$

Thus, in this superspace interpretation, s and \bar{s} are exterior derivative operators along the unphysical directions θ and $\bar{\theta}$ respectively.

The curvature two-form or Yang–Mills field strength associated with \mathcal{A} is defined in the usual fashion:

$$\mathcal{F} \equiv \tilde{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \tilde{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] . \quad (15)$$

From (9a), we can immediately make the identification

$$\mathcal{F} = F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda , \quad (16)$$

whence

$$\begin{aligned} F &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu , & \psi &= F_{\mu\theta} dx^\mu \wedge d\theta , \\ \phi &= \frac{1}{2} F_{\theta\theta} d\theta \wedge d\theta , & \bar{\psi} &= F_{\mu\bar{\theta}} dx^\mu \wedge d\bar{\theta} , \\ \bar{\phi} &= \frac{1}{2} F_{\bar{\theta}\bar{\theta}} d\bar{\theta} \wedge d\bar{\theta} , & \lambda &= F_{\theta\bar{\theta}} d\theta \wedge d\bar{\theta} . \end{aligned} \quad (17)$$

While F is the usual Yang–Mills curvature in the physical spacetime, the other five fields represent the curvature components along the various unphysical directions. Thus \mathcal{F} given by (9a) is the total Yang–Mills field strength in superspace. This is the geometrical interpretation of the fields occurring in topological Yang–Mills theory.

Indeed, if we define the super covariant derivatives

$$\begin{aligned} D &\equiv d + [A, \cdot] , \\ S &\equiv s + [c, \cdot] , \\ \bar{S} &\equiv \bar{s} + [\bar{c}, \cdot] , \end{aligned} \tag{18}$$

it can be readily shown, to our expectation, that

$$\begin{aligned} D^2 X &= [F, X] , & (SD + DS)X &= [\psi, X] , \\ S^2 X &= [\phi, X] , & (\bar{S}D + D\bar{S})X &= [\bar{\psi}, X] , \\ \bar{S}^2 X &= [\bar{\phi}, X] , & (S\bar{S} + \bar{S}S)X &= [\lambda, X] , \end{aligned} \tag{19}$$

for any field X . This is a pleasing consistency check.

The second equation of (9) may be thought of as an extended Bianchi identity in superspace that \mathcal{F} satisfies, that is

$$\tilde{D}\mathcal{F} \equiv \tilde{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0 . \tag{20}$$

Note that the two equations of (9) together imply that $\tilde{d}^2 = 0$, which yields the nilpotency condition (1), as well as $d^2 = 0$, $ds + sd = 0$, etc.

We can now recover ordinary Yang–Mills theory by imposing the so-called horizontality condition for \mathcal{F} [5]:

$$\mathcal{F} = F . \tag{21}$$

This is tantamount to requiring that the Yang–Mills field strength vanish along the unphysical directions, that is, the identical vanishing of the fields ψ , $\bar{\psi}$, ϕ , $\bar{\phi}$ and λ . This therefore is the very simple relationship between the geometries of topological and ordinary Yang–Mills theories. Indeed, it is just as trivial to proceed in the other direction. The horizontality condition was discovered many years ago, when people tried to understand the BRS-quantisation of ordinary Yang–Mills theory within the context of superspace. If one had then tried to generalise this to the case of non-vanishing Yang–Mills field strengths in the unphysical directions, one would have had at hand topological Yang–Mills theory. This intriguing historical alternative would have then resulted in the much earlier discovery of the topological theory. Incidentally, this argument also clearly shows that the topological theory, loosely speaking, is the “most general type” of Yang–Mills theory possible, in that there is no more room in superspace for any other direct generalisation.

To summarise, we have enlarged physical spacetime by adjoining two new but unphysical directions. In this superspace, we have introduced a generalised gauge potential \mathcal{A} and its field strength \mathcal{F} . These fields each consists of the classical component and extra ghost components coexisting together. The idea of having this unified treatment is so that the original gauge invariance of the classical field can just be relegated to invariant transformations of its ghost part in the unphysical directions. Hence, when one integrates the action over the physical subspace, there is no problem with zero modes of the classical fields.

For the case of the ordinary Yang–Mills theory, the gauge potential has a simple gauge invariance of the form $\delta A = D\Lambda$. It is made into translations in the unphysical directions of the form $sA = Dc$ and $\bar{s}A = D\bar{c}$, where c and \bar{c} are the ghost components of A . This caters for all the gauge invariance the theory possesses. So our final unified fields in superspace are $\mathcal{A} = A + c + \bar{c}$ and $\mathcal{F} = F$.

The case of topological Yang–Mills theory is slightly more complicated. The theory has, in addition to the normal gauge invariance above, a topological symmetry of the form $\delta A = \Lambda$. This may be regarded as an invariance of F , given by $\delta F = D\Lambda$. By the same process done with the A field, we push this gauge invariance of F to its ghost components. Since F is a two-form, it has five extra ghost components in all.

4. SU(3) triality

Our construction of the BRS and anti-BRS symmetry in topological Yang–Mills theory has also revealed the presence of a hidden symmetry otherwise absent in ordinary Yang–Mills theory. There seems to exist a strange type of SU(3) triality between the physical direction, the θ -direction and the $\bar{\theta}$ -direction of superspace. This can be seen from Fig. 1. The gauge potential triplet

$$\begin{array}{ccc} & A & \\ \bar{c} & & c \end{array}$$

traces out an isosceles triangle in the ghost number–form degree plane. This triplet may be taken to be the familiar weight diagram for the fundamental (1,0)-representation of SU(3), denoted by **3**. The gauge field strength sextet

$$\begin{array}{ccccc} & & F & & \\ & \bar{\psi} & & \psi & \\ \bar{\phi} & & \lambda & & \phi \end{array}$$

is also an isosceles triangle which may be regarded as the weight diagram for the (2,0)-representation of SU(3), denoted by **6**.

The relationship between these two representations, however, is rather vague to us. From $SU(3)$ representation theory, we have the relation

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6}_s \oplus \mathbf{\bar{3}}_a , \quad (22)$$

that is, the tensor product of two fundamental representations ($\mathbf{3}$) splits into symmetric ($\mathbf{6}$) and anti-symmetric ($\mathbf{\bar{3}}$) parts. This probably describes specifically the equations

$$\begin{aligned} dA + \frac{1}{2}[A, A] &= F , & sA + dc + [A, c] &= \psi , \\ sc + \frac{1}{2}[c, c] &= \phi , & \bar{s}A + d\bar{c} + [A, \bar{c}] &= \bar{\psi} , \\ \bar{s}\bar{c} + \frac{1}{2}[\bar{c}, \bar{c}] &= \bar{\phi} , & s\bar{c} + \bar{s}c + [c, \bar{c}] &= \lambda . \end{aligned} \quad (23)$$

If we disregard the exterior derivative terms for the moment, we may take the tensor product \otimes to be the graded bracket. Taking brackets of the triplet $\mathbf{3}$ then yields the sextet $\mathbf{6}$ of fields. The anti-commuting triplet $\mathbf{\bar{3}}$ vanishes because the graded brackets occurring here are anti-commutators, and thus only single out the symmetric parts. Perhaps there is a way to incorporate the action of the exterior derivatives into the definition of the tensor product, but we will not attempt it here.

Let us just mention another curiosity of Fig. 1. Recall that there we briefly introduced the operators \mathcal{D} , \mathcal{S} and $\bar{\mathcal{S}}$. Explicitly, we could set

$$\mathcal{D} \equiv d + [A, \cdot] ; \quad (24)$$

unless when acting on odd fields (A, c, \bar{c}) , in which case we take

$$\mathcal{D} \equiv d + \frac{1}{2}[A, \cdot] . \quad (25)$$

In a similar manner, we set

$$\begin{aligned} \mathcal{S} &\equiv s + [c, \cdot] , & \text{or} & & \mathcal{S} &\equiv s + \frac{1}{2}[c, \cdot] ; \\ \bar{\mathcal{S}} &\equiv \bar{s} + [\bar{c}, \cdot] , & \text{or} & & \bar{\mathcal{S}} &\equiv \bar{s} + \frac{1}{2}[\bar{c}, \cdot] . \end{aligned} \quad (26)$$

In this notation, observe that our field equations (3), (5), (7) and (8) may be compactly rewritten as

$$\begin{aligned} \mathcal{D}A &= F , & \mathcal{D}F &= 0 , & \mathcal{S}F + \mathcal{D}\psi &= 0 , \\ \mathcal{S}c &= \phi , & \mathcal{S}\phi &= 0 , & \bar{\mathcal{S}}F + \mathcal{D}\bar{\psi} &= 0 , \\ \bar{\mathcal{S}}\bar{c} &= \bar{\phi} , & \bar{\mathcal{S}}\bar{\phi} &= 0 , & \mathcal{D}\phi + \mathcal{S}\psi &= 0 , \\ \mathcal{D}c + \mathcal{S}A &= \psi , & & & \mathcal{D}\bar{\phi} + \bar{\mathcal{S}}\bar{\psi} &= 0 , \\ \mathcal{D}\bar{c} + \bar{\mathcal{S}}A &= \bar{\psi} , & & & \mathcal{S}\lambda + \bar{\mathcal{S}}\phi &= 0 , \\ \mathcal{S}\bar{c} + \bar{\mathcal{S}}c &= \lambda , & & & \bar{\mathcal{S}}\lambda + \mathcal{S}\bar{\phi} &= 0 , \\ & & & & \mathcal{D}\lambda + \mathcal{S}\bar{\psi} + \bar{\mathcal{S}}\psi &= 0 . \end{aligned} \quad (27)$$

It can be easily recognised that the first column of six equations represents the curvature equation (9a), while the other two columns describe the Bianchi identity (9b). These equations have an attractive pictorial representation in Fig. 1. The triplet (A, c, \bar{c}) is considered fundamental, out of which all the other fields are constructed from. Taking one of the sextet fields $(F, \psi, \bar{\psi}, \phi, \bar{\phi}, \lambda)$, it may be expressed as the sum of its four adjacent fields, each of which is acted upon by one of the operators $(\mathcal{D}, \mathcal{S}, \bar{\mathcal{S}})$. For example, ψ lies next to the fields A and c . Hence, it may be written as $\psi = \mathcal{S}A + \mathcal{D}c$. This accounts for the first column.

The other two columns express the fact that all other fundamental fields outside the triplet and sextet vanish. Take for example, the position in Fig. 1 with ghost number +1 and form degree 2. It may be written, by the procedure outlined above, as $\mathcal{S}F + \mathcal{D}\psi$. But then by (27), this field is identically zero.

It is not clear to us whether or not the simplicity of the equations in (27) is trying to tell us something else. For example, while $\mathcal{D}^2 = 0$, the equations do not seem to imply that $\mathcal{S}^2 = \mathcal{S}\bar{\mathcal{S}} + \bar{\mathcal{S}}\mathcal{S} = \mathcal{S}\mathcal{D} + \mathcal{D}\mathcal{S} = \dots = 0$, despite first appearances. Is it possible to construct from this some sort of superspace version of (gauge covariant) de Rham cohomology? Perhaps we should not take (27) too seriously in the first place, as the dual meanings of \mathcal{D} , \mathcal{S} and $\bar{\mathcal{S}}$ may be rather misleading.

5. Recovery of standard results

We have thus so far in this paper, built the solid foundations of the BRS and anti-BRS symmetry into topological Yang–Mills theory, and ended with a few speculative remarks. We will now concentrate, in the rest of this paper, on reproducing the work of Baulieu and Singer [8] using our new formalism.

Let us define the following auxiliary fields: b , an even scalar field with vanishing ghost number; κ , an odd one-form with vanishing ghost number; an odd scalar field η with ghost number one; and its corresponding anti-ghost $\bar{\eta}$. Considering the four equations in (7) and (8), we set

$$\begin{aligned}
s\bar{c} &= b, & \bar{s}c &= \lambda - b - [c, \bar{c}], \\
s\bar{\psi} &= -\kappa, & \bar{s}\psi &= -D\lambda + \kappa - [c, \bar{\psi}] - [\bar{c}, \psi], \\
s\lambda &= \eta, & \bar{s}\lambda &= -\bar{\eta} - [\bar{c}, \lambda] - [c, \bar{\phi}], \\
s\bar{\phi} &= \bar{\eta}, & \bar{s}\phi &= -\eta - [\bar{c}, \phi] - [c, \lambda].
\end{aligned} \tag{28}$$

Of course, we could have made a more symmetrical choice of these transformations, but it does not really matter in our later considerations.

To ensure the continued nilpotency of s and \bar{s} , we have to derive the appropriate s and \bar{s} transformations on the auxiliary fields. Clearly,

$$sb = s\kappa = s\eta = s\bar{\eta} = 0 . \quad (29)$$

However, \bar{s} acting on these fields is more complicated because of our choice of asymmetry in (28), and we will not write them down here.

Thus, we have demonstrated how the missing partner of $\bar{\eta}$ in the BRS formalism naturally appears when we include the anti-BRS symmetry. We have also managed to reproduce the first and third row of equations in (4), within our BRS and anti-BRS formalism. What about the second row of equations, involving the fields $\bar{\chi}$ and B ? Recall that in ref. [8], the two-form $\bar{\chi}$ is regarded as the anti-ghost of ψ . But in our analysis of anti-BRS symmetry, the actual anti-ghost is the one-form $\bar{\psi}$. Luckily, the reconciliation of this discrepancy is fairly obvious; these two fields are related by

$$\bar{\chi} = d\bar{\psi} , \quad (30a)$$

and similarly for the auxiliary fields:

$$B = d\kappa . \quad (30b)$$

(By our choice of gauge-fixing conditions later, $\bar{\chi}$ and B will be both self-dual two-forms.) While the choice of these relationships is not unique, it will become apparent later why we have made the most natural choice. Thus, the second row of equations in (4) follows from our analysis as well.

Hence, we have explained the few questions raised earlier, that is on why the anti-ghost $\bar{\chi}$ of ψ is not a one-form; and on the missing partner of $\bar{\eta}$. This happily demonstrates the conceptual power and beauty of our combined BRS and anti-BRS approach.

Our final task is to derive the complete gauge-fixed quantum action of topological Yang–Mills theory as written down in ref. [8]. To do so, it is useful to first translate our s and \bar{s} transformations (3), (5) and (28), from differential forms into tensor notation. This is an exercise left to the reader. The total gauge-fixed action consists of the classical action (2) plus an s - and \bar{s} -exact part. It is of the form

$$\int_M d^4x \operatorname{tr} \left[F_{\mu\nu} * F^{\mu\nu} + s\bar{s} \{ \dots \} \right] , \quad (31)$$

for any choice of appropriate gauge-fixing terms within the curly brackets (with vanishing ghost number). The resulting quantum action will then be s - and \bar{s} -invariant, because of the nilpotency condition (1). Recall that $*$ is the duality operator, given by $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

With this quantum action I , we can define the partition function

$$Z = \int \mathcal{D}X \exp(-I/e^2) , \quad (32)$$

where $\mathcal{D}X$ denotes the path integral over the appropriate fields present in I , and e is the coupling constant. Witten [6] has showed how this and suitable correlation functions of it generate the Donaldson invariants of smooth four-manifolds. Also recall that because topological field theories are generally independent of the value of the coupling constant e , we can take the semi-classical limit of very small e [6]. In this case, only the quadratic terms of I are retained, and any higher-order terms drop out. This means that we can ignore the bracket terms in our s and \bar{s} transformations.

Now recall that the s -exact gauge-fixing part of Baulieu and Singer's [8] action has the form

$$s \{ \bar{\chi}_{\mu\nu} (F^{\mu\nu} \pm *F^{\mu\nu}) \pm \frac{1}{2}\rho\bar{\chi}_{\mu\nu}B^{\mu\nu} + \bar{\phi}\partial_\mu\psi^\mu + \bar{c}\partial_\mu A^\mu + \frac{1}{2}\sigma\bar{c}b \} , \quad (33)$$

where ρ and σ are arbitrary real gauge parameters. But observe that

$$\begin{aligned} s\bar{s} \{ \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}A_\mu A^\mu + \bar{\psi}_\mu\psi^\mu \} \\ = s \{ 2\partial_{[\mu}\bar{\psi}_{\nu]}F^{\mu\nu} + \bar{\phi}\partial_\mu\psi^\mu + \bar{c}\partial_\mu A^\mu + \bar{\psi}_\mu(\partial^\mu\lambda - \kappa^\mu - A^\mu) \} . \end{aligned} \quad (34)$$

After the field redefinition

$$\partial^\mu\lambda \rightarrow \partial^\mu\lambda + \kappa^\mu + A^\mu , \quad (35)$$

we have

$$\begin{aligned} s\bar{s} \{ \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}A_\mu A^\mu + \bar{\psi}_\mu\psi^\mu \} \\ = s \{ 2\partial_{[\mu}\bar{\psi}_{\nu]}F^{\mu\nu} + \bar{\phi}\partial_\mu\psi^\mu + \bar{c}\partial_\mu A^\mu - \lambda\partial_\mu\bar{\psi}^\mu \} . \end{aligned} \quad (36)$$

From this expression, let us now make the observation that the A_μ gauge-fixing condition is $\partial_\mu A^\mu = 0$; and that the ψ_μ gauge-fixing condition is $\partial_\mu\psi^\mu = 0$. The apparent gauge-fixing condition for $F_{\mu\nu}$ is $\partial_\mu F^{\mu\nu} = 0$, but it is a bad choice. This is because it is just the Bianchi identity modulo a higher-order term, and can be made always true

in this context. Instead, the usual choice of gauge for $F_{\mu\nu}$ is either the self-duality or anti-self-duality condition imposed on it. This is set by replacing (34) with

$$\begin{aligned} \text{s}\bar{\text{s}} \left\{ \frac{1}{2} F_{\mu\nu} P_{\pm} F^{\mu\nu} - \frac{1}{2} A_{\mu} A^{\mu} + \bar{\psi}_{\mu} \psi^{\mu} \right\} \\ = \text{s} \left\{ 2\partial_{[\mu} \bar{\psi}_{\nu]} P_{\pm} F^{\mu\nu} + \bar{\phi} \partial_{\mu} \psi^{\mu} + \bar{c} \partial_{\mu} A^{\mu} - \lambda \partial_{\mu} \bar{\psi}^{\mu} \right\} , \end{aligned} \quad (37)$$

where P_{\pm} is the (anti-) self-dual projection operator given by $P_{\pm} \equiv \frac{1}{2}(1 \pm *)$. Enforcing the gauge $P_{\pm} F_{\mu\nu} = 0$ is the “anti-ghost” field $\partial_{[\mu} \bar{\psi}_{\nu]}$, which we may conveniently rename $\bar{\chi}_{\mu\nu}$. It is an anti-symmetric and (anti-) self-dual rank-two tensor. At the same time, we set $B_{\mu\nu} = \partial_{[\mu} \kappa_{\nu]}$.

Observe now that (37) is very nearly the same as (33), but for the choice of gauge $\rho = \sigma = 0$. There is, however, one extra term

$$\text{s} \left\{ -\lambda \partial_{\mu} \bar{\psi}^{\mu} \right\} = \lambda \partial_{\mu} \kappa^{\mu} - \eta \partial_{\mu} \bar{\psi}^{\mu} , \quad (38)$$

that does not occur in the latter equation. Fortunately, the path integrals over the fields η and λ yield delta functions which enforce the conditions that

$$\partial_{\mu} \bar{\psi}^{\mu} = 0 , \quad \partial_{\mu} \kappa^{\mu} = 0 . \quad (39)$$

Note that these do not affect the definitions of $\bar{\chi}_{\mu\nu}$ and $B_{\mu\nu}$. Hence, we arrive at the gauge-fixed action of Baulieu and Singer [8], up to negligible higher-order terms.

The astute reader would notice that in making the field redefinitions (30), we are changing the measure of the path integral by Jacobian-like terms. Symbolically, these changes are

$$\begin{aligned} \mathcal{D}\bar{\psi} &= \mathcal{D}\bar{\chi} \left[\det \frac{\mathcal{D}\bar{\psi}}{\mathcal{D}\bar{\chi}} \right]^{-1} , \\ \mathcal{D}\kappa &= \mathcal{D}B \left[\det \frac{\mathcal{D}\kappa}{\mathcal{D}B} \right] , \end{aligned} \quad (40)$$

where there is an extra inverse in the first Jacobian because $\bar{\psi}_{\mu}$ and $\bar{\chi}_{\mu\nu}$ are anti-commuting fields. From the relations in (30), observe that the two Jacobians cancel each other. Hence the measure of the path integral remains the same even after our change of variables.

6. Concluding remarks

By standard arguments, it can be shown that from the gauge symmetry of a classical theory, one can always build up the corresponding BRS and anti-BRS symmetry of the quantum theory counterpart. Thus, in this paper, we have studied the quantisation of topological Yang–Mills theory, from the point of view of both the BRS and anti-BRS symmetry. This procedure explains the various peculiarities that occur in previous treatments, which consider only the BRS symmetry. In particular, we have resolved the issue of why the anti-ghost of the ghost vector field ψ is not a vector field.

Another conceptual advantage of our approach is that it gives a beautiful geometrical interpretation of the ghost and anti-ghost fields occurring in our quantisation process. They turn out to be the connection and curvature components in the two unphysical, anti-commuting directions of superspace. In particular, ordinary Yang–Mills theory is recovered by imposing the condition that these curvature components vanish. We have also uncovered a certain triality between these two unphysical directions of superspace, and the physical direction itself. This triality seems to be described by the Lie group $SU(3)$.

Finally, we showed how to recover the standard gauge-fixed topological Yang–Mills action from our formalism. We do not claim that our treatment simplifies this gauge-fixing procedure, as it clearly does not! Instead, our aim in this paper has been to demonstrate how to incorporate the anti-BRS symmetry into topological Yang–Mills theory, and highlight the power of this method in revealing the elegant geometry and symmetries of the theory.

One is entitled to ask whether the presence of this extra anti-BRS symmetry could be used to modify Witten’s topological Yang–Mills theory in any way. Indeed, it is easy to write down the generalised descent equation [8] which includes the \bar{s} operator:

$$(d + s + \bar{s})(F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda)^n = 0 , \quad (41)$$

where n is an integer greater than or equal to 2. By expanding this equation out in form degree and ghost number, it is in principle possible to construct new observables of topological Yang–Mills theory with the appropriate ghost number (see for example, sec. 5.2.7 of ref. [7]).

Let us point out another possible extension of this work. As we have seen, the BRS and anti-BRS symmetry can be interpreted in terms of connections and curvatures in superspace. It would be very pleasing if we could write the gauge-fixed action, (31) with

(37), entirely and covariantly in terms of superfields, like \mathcal{A} and \mathcal{F} introduced earlier. If done, this would no doubt simplify the action and make any symmetries of the theory more manifest. We do not attempt it here however, because no entirely satisfactory superfield formulation yet exists even for ordinary Yang–Mills theory. Recent and interesting attempts involving the ordinary Yang–Mills theory may be found in refs. [11] and [12]. Attempts to describe topological Yang–Mills theory in terms of superfields have been made in refs. [13] and [14]. To find a complete superfield formalism would surely be an interesting exercise for the motivated reader.

Finally, we should mention that anti-BRS symmetry in topological Yang–Mills theory has also been considered very recently in ref. [15]. However, the structure of their symmetry is very different from ours. In particular, they have constructed their BRS and anti-BRS symmetry so that it reproduced the equations (3), where the s operator consists of their BRS and anti-BRS operators added together. By contrast, our approach assumes that (3) alone characterises the BRS part, and there exists a separate anti-BRS part as in (5). Indeed, we have followed what is usually done in ordinary Yang–Mills theory [5]. The reader is invited to compare and contrast the two approaches.

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